EUCLIDEAN AND NON-EUCLIDEAN GEOMETRY - WEEK 1

MICHAEL ALBANESE

Philosophical Question: Is mathematics invented or discovered?

Our goal will be to study geometry from a more rigorous perspective. In order to achieve this goal, first we will explore how this rigorous perspective applies in a simpler setting, the real numbers.

1. Properties of the real numbers

The collection of real numbers have a notion of addition ($+$) and multiplication ($\times$) satisfying

(1) $a+(b+c)=(a+b)+c$ and $a\times(b\times c)=(a\times b)\times c$ (associativity) *There's a choice to be made when adding or multiplying three numbers: which pair do we add/multiply first? Associativity tells us that we get the same result, regardless of the choice we make.*

(2) $a+b=b+a$ and $a\times b=b\times a$ (commutativity) *It doesn't matter the order in which we perform the addition or subtraction.*

(3) there is an element 0 such that $a+0=a$ (additive identity)

(4) for every a, there is b such that $a+b=0$ (additive inverse)

(5) there is an element $1\neq 0$ such that $a\times 1=a$ (multiplicative identity)

(6) for every $a\neq 0$, there is b such that $a\times b=1$ (multiplicative inverse)

(7) $a\times(b+c)=a\times b+a\times c$ (distributivity of multiplication over addition).

All of elementary arithmetic (except inequalities) follows from these properties[1]. What are some things from arithmetic that don't appear in this list? Subtraction and division. How are they related to addition and multiplication? Need to be able to talk about $-a$ and a^{-1}.

Theorem 1.1. *Every element has a unique additive inverse. That is, if $a+b=0$ and $a+c=0$, then $b=c$.*

Proof. Consider $(a+b)+c$. On the one hand, we have

$$\begin{aligned}(a+b)+c &= 0+c && \text{hypothesis}\\ &= c+0 && (2)\\ &= c && (3)\end{aligned}$$

On the other hand,

$$\begin{aligned}(a+b)+c &= a+(b+c) && (1)\\ &= a+(c+b) && (2)\end{aligned}$$

[1]Two other properties must be included in order to discuss inequalities. Namely, there exists a collection P of positive numbers such that

(8) for every a, exactly one of the following hold: a belongs to P, $-a$ belongs to P, or $a=0$ (trichotomy)*Note, this requires $-a$ to be defined first.*

(9) if a and b belong to P, then so does $a+b$ and $a\times b$ (closure under addition and multiplication).

$$
\begin{aligned}
&= (a + c) + b && (1) \\
&= 0 + b && \text{hypothesis} \\
&= b + 0 && (2) \\
&= b && (3)
\end{aligned}
$$

Therefore $b = (a + b) + c = c$. □

Definition 1.2. For any a, we denote the unique additive inverse by $-a$.

Definition 1.3. For any a and b, their difference, denoted $a - b$, is defined to be $a + (-b)$.

Exercise 1.4. Prove that if $a \neq 0$, then a has a unique multiplicative inverse (compare with the proof of Theorem 1.1).

Definition 1.5. For any $a \neq 0$, we denote the unique multiplicative inverse by a^{-1}.

Definition 1.6. For any a and b with $b \neq 0$, their quotient, denoted $a \div b$, is defined to be $a \times (b^{-1})$.

What if $b = 0$? Why can't we divide by 0? Are we not allowed to? According to the above definition, we would need a multiplicative inverse for 0. Note that such an element is not guaranteed by (6), but that doesn't necessarily mean it doesn't exist.

Theorem 1.7. *0 has no multiplicative inverse. More precisely, $0 \times b = 0$ for every b.*

Proof. Note that $0 \times b = b \times 0$ by (2), so it is enough to show that $b \times 0 = 0$. Consider $b \times (0 + 0)$.

We can use (7) to give $b \times (0 + 0) = b \times 0 + b \times 0$.

On the other hand, we have $0 + 0 = 0$ by (3), so $b \times (0 + 0) = b \times 0$.

Therefore

$$b \times 0 + b \times 0 = b \times (0 + 0) = b \times 0.$$

By (5), we can form $-(b \times 0)$. Adding this to both sides gives:

$$
\begin{aligned}
(b \times 0 + b \times 0) - (b \times 0) &= b \times 0 - (b \times 0) && \\
b \times 0 + (b \times 0 - b \times 0) &= 0 && \text{(1) and Definition 1.3} \\
b \times 0 + 0 &= 0 && \\
b \times 0 &= 0 && (3)
\end{aligned}
$$

□

Because 0 has no multiplicative inverse, we cannot extend Definition 1.6 to allow for division by 0. What does 0^{-1} mean then? Nothing! It is undefined, whereas for $a \neq 0$, the expression a^{-1} is defined (by Definition 1.5). This is no different than what we experience with language. Every word is just a string of letters, but not every string of letters is a word. A string of letters becomes a word when we give it a definition. For example, bag, beg, big, bog, and bug are all words, while bzg is not - it is undefined, while all the others are defined (in the dictionary). Similarly, every meaningful mathematical expression is a string of symbols, but not every string of symbols is a meaningful mathematical expression.

Other theorems that can be proved using (1) – (7):

- $-a = (-1) \times a$,

- if $a \times b = 0$, then $a = 0$ or $b = 0$,
- $-0 = 0$,
- $1^{-1} = 1$,
- $-(-a) = a$,
- $(-a) \times b = a \times (-b) = -(a \times b)$, and
- $(a^{-1})^{-1} = a$ for $a \neq 0$.

Note, the collection of real numbers is not the only one which satisfies (1) – (7). For example, the rational numbers also do (the sum and product of rational numbers is rational, 0 and 1 are rational, so only need to check that if a is rational, so is a, and if a is a non-zero rational, then so too is a^{-1}). What about the integers (whole numbers)? They don't satisfy (6): there is no integer b such that $2 \times b = 1$. Another collection which does satisfy (1) – (7) is the set of numbers of the form $p + q\sqrt{2}$ where p and q are rational (the sum and product of numbers of this form are again of this form, 0 and 1 are of this form, and the additive and multiplicative inverses of such a number is also of this form - for the multiplicative inverse, multiply by the conjugate).

If we adjust (1) – (7) to allow for different notions of addition and multiplication (and different choices of additive and multiplicative identities), then there are many more examples, some of which have no obvious relationship to the real numbers.

Example 1.8. Consider only the numbers 0, 2, 4, 6, and 8. We define a new addition and multiplication for these numbers, denoted by $\boldsymbol{+}$ and $\boldsymbol{\times}$ respectively, as follows: do the usual addition or multiplication and only use the last digit. For example, $4 \boldsymbol{+} 6 = 0$ because $4+6 = 1\underline{0}$, and $4 \boldsymbol{\times} 6 = 4$ because $4 \times 6 = 2\underline{4}$.

Exercise 1.9. (a) Complete the following tables:

$\boldsymbol{+}$	0	2	4	6	8
0					
2					
4					
6					
8					

$\boldsymbol{\times}$	0	2	4	6	8
0					
2					
4					
6					
8					

(b) What is the additive identity? Identify $\boldsymbol{-}0$, $\boldsymbol{-}2$, $\boldsymbol{-}4$, $\boldsymbol{-}6$, and $\boldsymbol{-}8$.

(c) What is the multiplicative identity? Identify $a^{\boldsymbol{-1}}$ for those a other than the additive identity.

(d) With this notion of addition and mutliplication, what is $\dfrac{8^{\boldsymbol{-1}} \boldsymbol{-} 4^{\boldsymbol{-1}}}{2 \boldsymbol{\times} 4 \boldsymbol{+} 2 \boldsymbol{\times} 2}$?

Example 1.10. Let (a, b) be an ordered pair of real numbers. We define addition of such pairs as $(a, b) \boldsymbol{+} (c, d) := (a + c, b + d)$ and multiplication by $(a, b) \boldsymbol{\times} (c, d) = (a \times c - b \times d, a \times d + b \times c)$. For example,

$$\begin{aligned}
(4, 6) \boldsymbol{+} (-1, 2) &= (4 + (-1), 6 + 2) \\
&= (3, 8) \\
(4, 6) \boldsymbol{\times} (-1, 2) &= (4 \times (-1) - 6 \times 2, 4 \times 2 + 6 \times (-1)) \\
&= (-4 - 12, 8 - 6) \\
&= (-16, 2).
\end{aligned}$$

Exercise 1.11. (a) What is the additive identity? Identify $-(a, b)$.

(b) (Harder) What is the multiplicative identity? Identify $(a, b)^{-1}$ for (a, b) not equal to the additive identity.

(c) Show that $(3, 4) \times (3, -4) = (5, 0) \times (5, 0)$ and $(5, 12) \times (5, -12) = (13, 0) \times (13, 0)$.

There may not be numbers involved at all.

Example 1.12. Consider the collection of four elements $\{A, B, C, D\}$ with addition and multiplication given by

$+$	A	B	C	D
A	C	D	A	B
B	D	C	B	A
C	A	B	C	D
D	B	A	D	C

$\times$	A	B	C	D
A	B	D	C	A
B	D	A	C	B
C	C	C	C	C
D	A	B	C	D

Exercise 1.13. (a) What is the additive identity? Identify $-A$, $-B$, $-C$, and $-D$.

(b) What is the multiplicative identity? Identify a^{-1} for those a other than the additive identity.

(c) With this notion of addition and mutliplication, what is $\dfrac{D^{-1} - A^{-1}}{A \times B + C \times D}$?

Important Question: What was the point of all this?

2. Axiomatic Systems

An *axiomatic system* consists of the following:

- *primitive terms*: technical words that are used in the axioms without formal definitions (e.g., collection, element);
- *defined terms*: other technical terms that are given precise, unambiguous definitions in terms of the primitive terms and other previously defined terms (e.g., additive identity, multiplicative identity);
- *axioms* (also called *postulates*): mathematical statements about the primitive and defined terms that will be assumed to be true without proof (e.g., (1) – (7));
- *theorems*: mathematical statements about the primitive and defined terms that can be given rigorous proofs based only on the axioms, definitions, previously proved theorems, and rules of logic (e.g. the theorems mentioned previously).

The axioms (1) – (7) are called the axioms of a *field*. So the real numbers, the rational numbers, the numbers of the form $p + q\sqrt{2}$ where p and q are rational, as well as Examples 1.8, 1.10, and 1.12, are all examples of fields. All the theorems have proved and can prove about the real numbers using (1) – (7) apply in exactly the same way to any field because we only used statements which are true in any field. For example, Theorem 1.7 is true for every field (where 0 is replaced with the additive identity). In Example 1.12, the additive identity is C and indeed, multiplication with C always gives C as can be seen in the multiplication table.

University of Waterloo

Email address: `m3albane@uwaterloo.ca`